E9 203 Compressed Sensing & Sparse SP

16 Apr. 2021.

Last time:

Recovery of individual sparse vecs.

$$\mathrm{sgn}(z) = \begin{cases} \frac{z}{|z|} & z \neq 0 \\ 0 & z = 0. \end{cases}$$

Thm. 4.26

Given $A \in \mathbb{C}^{m \times N}$, the vec. $x \in \mathbb{C}^N$ with $\mathrm{supp}(x) = S$ is the unique min. of $\|z\|_1$ s.t. $Az = Ax$ if one of the foll. equivalent conditions hold:

(a) $\left| \sum_{j \in S} \overline{\mathrm{sgn}(x_j)} v_j \right| < \|v_{\bar{S}}\|_1 \quad \forall\, v \in \mathcal{N}(A) \setminus \{\underline{0}\}$

(b) A_S is injective, and $\exists\, h \in \mathbb{C}^m$ s.t.

$$(A^H h)_j = \mathrm{sgn}(x_j), \quad j \in S$$

$$|(A^H h)_\ell| < 1, \quad \ell \in \bar{S}.$$

Today: · A few remarks about theorem 4.26.
· Characterization via tangent cones.

Remarks:

- If $x \in \mathbb{C}^N$ satisfies (a) of Thm. 4.26, then all vecs. $x' \in \mathbb{C}^N$ w/ $\mathrm{supp}(x') = S' \subset S$ & $\mathrm{sgn}(x')_{S'} = \mathrm{sgn}(x)_{S'}$, are also exactly recovered via BP.
- Thm. 4.26 can be made stable (compressible vecs) and robust (additive noise); see Ex. 4.17 (HW).
- The converse to Thm. 4.26 does not hold in the complex setting, but it does hold in the real-valued case. See Thm. 4.30.

Stable & robust version of the result:

Thm. 4.33

Let $A = [a_1, \ldots, a_N] \in \mathbb{C}^{m \times N}$, let $x \in \mathbb{C}^N$ with its s largest values supported on S. Let $y = Ax + e$, $\|e\|_2 \leq \eta$. For $\delta, \beta, \gamma, \theta, \tau \geq 0$, $\delta < 1$, assume

$$|||A_S^H A_S - I|||_2 \leq \delta, \quad \max_{\ell \in \bar{S}} \|A_S^H a_\ell\|_2 \leq \beta$$

and $\exists\, u = A^H h \in \mathbb{C}^N$ with $h \in \mathbb{C}^m$ s.t.

$$\|u_S - \mathrm{sgn}(x_S)\|_2 \leq \gamma, \quad \|u_{\bar{S}}\|_\infty \leq \theta, \quad \|h\|_2 \leq \tau\sqrt{s}.$$

If $\rho \triangleq \theta + \frac{\beta\gamma}{1-\delta} < 1$, then a minimizer $x^\#$ of $\|z\|_1$ s.t. $\|Az - y\|_2 \leq \eta$ satisfies:

$$\|x - x^\#\|_2 \leq C_1\, \sigma_s(x)_1 + (C_2 + C_3\sqrt{s})\,\eta$$

$\|x - x^\#\|_2 \leq \frac{C_1}{\sqrt{s}} \sigma_s(x)_1 + C_2 \eta$

for some consts $C_1, C_2, C_3 > 0$ dep. on $\delta, \beta, \gamma, \theta, \tau$ as

$$\begin{cases} C_1 = \frac{2}{1-\rho}\left(1 + \frac{\beta}{1-\delta}\right); \quad C_3 = \frac{2\tau}{1-\rho}\left(1 + \frac{\beta}{1-\delta}\right) \\ C_2 = \frac{2\sqrt{1+\delta}}{1-\delta}\left[\left(\frac{\gamma}{1-\rho}\right)\left(1 + \frac{\beta}{1-\delta}\right) + 1\right]. \end{cases}$$

Remark: $\delta = \beta = \gamma = \frac{1}{2}$, $\theta = \frac{1}{4}$, $\tau = 2$

$\rho = \frac{3}{4}$, $C_1 = 16$, $C_2 = 10\sqrt{6}$, $C_3 = 32$.

Next Characterization: Tangent cones to ℓ_1 balls:

Given $x \in \mathbb{R}^N$, define the convex cone

$$T(x) = \mathrm{cone}\left(\{z - x : z \in \mathbb{R}^N, \|z\|_1 \leq \|x\|_1\}\right)$$

where cone (T) is the smallest convex cone containing T. It is defined as:

$$\mathrm{cone}(T) = \left\{ \sum_{j=1}^{n} t_j x_j : n \geq 1,\ t_1 \ldots t_n \geq 0,\ x_1 \ldots x_n \in T \right\}.$$

Thm. 4.35

For $A \in \mathbb{R}^{m \times N}$, $x \in \mathbb{R}^N$ is the unique minimizer of $\|z\|_1$ s.t. $Az = Ax$ iff $\mathcal{N}(A) \cap T(x) = \{\underline{0}\}$.

Proof: Suppose $\mathcal{N}(A) \cap T(x) = \{\underline{0}\}$.

Let $x^\#$ be an ℓ_1 minimizer.

$\|x^\#\|_1 \leq \|x\|_1$ & $Ax^\# = Ax$.

$\Rightarrow v \triangleq x^\# - x, \quad v \in T(x) \cap \mathcal{N}(A) = \{\underline{0}\}$

$\Rightarrow x^\# = x$, i.e., x is the unique ℓ_1 minimizer.

Conversely, suppose x is the unique ℓ_1 minimizer.

$v \in T(x) \setminus \{0\} \Rightarrow v = \sum t_j (z_j - x)$, $t_j \geq 0$, $\|z_j\|_1 \leq \|x\|_1$

$v \neq 0 \Rightarrow \sum t_j > 0 \Rightarrow$ consider $t_j' = \frac{t_j}{\sum t_i}$.

If $v \in \mathcal{N}(A)$, $A\left(\sum t_j' z_j\right) = Ax \quad (\because \sum t_j' = 1)$

while $\|\sum t_j' z_j\|_1 \leq \sum t_j' \|z_j\|_1 \leq \|x\|_1$

By uniqueness of the ℓ_1-min., this $\Rightarrow \sum t_j' z_j = x$.

$\Rightarrow v = 0$, a contradiction. Hence,

$$(T(x) \setminus \{0\}) \cap \mathcal{N}(A) = \emptyset$$

or

$T(x) \cap \mathcal{N}(A) = \{0\}$, which completes the proof. ☒

Can extend to robust recovery:

Thm. 4.36

$A \in \mathbb{R}^{m \times N}$, $x \in \mathbb{R}^N$, $y = Ax + e \in \mathbb{R}^m$ with $\|e\|_2 \leq \eta$. If

$$\inf_{\substack{v \in T(x) \\ \|v\|_2 = 1}} \|Av\|_2 \geq \tau \quad \text{for some } \tau > 0,$$

then, a minimizer $x^\#$ of $\|z\|_1$ s.t. $\|Az - y\|_2 \leq \eta$ satisfies $\|x - x^\#\|_2 \leq \frac{2\eta}{\tau}$.

Proof: If $x^\# = x$, nothing to prove. So suppose $x^\# \neq x$.

Then, $\|x^\#\|_1 \leq \|x\|_1 \Rightarrow v \triangleq \frac{x^\# - x}{\|x^\# - x\|_2} \in T(x)$.

$\|v\|_2 = 1 \Rightarrow \|Av\|_2 \geq \tau$ by the assump.

$\Rightarrow \|A(x^\# - x)\|_2 \geq \tau \|x^\# - x\|_2$

But $\|A(x^\# - x)\|_2 \leq \|Ax^\# - y\|_2 + \|y - Ax\|_2 \leq 2\eta$

$\therefore \|x^\# - x\| \leq 2\eta$ ☒

Hence $\| x - x^{\sharp} \|_2 \leq \frac{[illegible]}{\tau}$ [illegible]

<u>Remark</u>: Thm. 4.35, 4.36 extend the above results to the complex case, by defining

$$T(x) = \text{cone}\left(\{ z - x : z \in \mathbb{C}^N, \|z\|_1 \leq \|x\|_1 \} \right).$$

Low rank matrix recovery

$$X \in \mathbb{C}^{n_1 \times n_2}, \quad \text{rank}(X) \leq r$$

$$y = \mathcal{A}(X) \in \mathbb{C}^m$$

$\mathcal{A}$ ↳ linear map $\mathbb{C}^{n_1 \times n_2} \to \mathbb{C}^m$

Equivalent of (P_0) problem:

$$\min_{Z \in \mathbb{C}^{n_1 \times n_2}} \text{rank}(Z) \quad \text{s.t.} \quad \mathcal{A}(Z) = y$$

This is NP hard (Ex. 2.11).

$$\text{rank}(Z) = \| \underline{\sigma}(Z) \|_0 \quad \text{where} \quad \underline{\sigma}(Z) = \begin{bmatrix} \sigma_1(Z) \\ \vdots \\ \sigma_n(Z) \end{bmatrix}$$

$$n = \min(n_1, n_2)$$

is a vector containing the singular values of Z.

Following the ℓ_1-min. philosophy, relax the pb. to:

$$(P_*) : \min_{Z \in \mathbb{C}^{n_1 \times n_2}} \|Z\|_* \quad \text{s.t.} \quad \mathcal{A}(Z) = y$$

$\| \cdot \|_*$ is the nuclear norm:

$$\|Z\|_* \triangleq \sum_{j=1}^{n} \sigma_j(Z), \quad n = \min(n_1, n_2).$$

(ℓ_1-norm of the vec. of singular values).

<u>Properties</u>:

1. $\| \cdot \|_*$ is a norm (Appendix A.25)
2. (P_*) is a convex opt. pb.
3. Actually equivalent to an SDP.